III Geometric group theory – Example Sheet 3

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1. Let L be a graph isometric to $\mathbb{R}$, and consider a group G acting faithfully and cocompactly on L. Draw the possible quotient graphs of groups $G\backslash L$. Classify the possible such groups G.

2. Consider the fundamental groups of Dehn's homology spheres

$$G_n = \pi_1(M_n) = \langle x, y, z \mid x^2 z^{-1}\ ,\ y^3 z^{-1}\ ,\ (xy)^{6n+5} z^{-5n-4} \rangle$$

for $n \geq 0$. Suppose that G_n acts on a tree T.

   (a) Assume that z acts elliptically on T, and let $T' = \mathrm{Fix}(z)$. Prove that G_n preserves T', and deduce that G_n has a global fixed point.

   (b) Assume that z acts hyperbolically on T, and let $L = \mathrm{Axis}(z)$. Prove that G_n preserves T', and deduce that G_n surjects a cocompact group of isometries of L.

   (c) Conclude that G_n has Serre's Property FA.

3. Consider a matrix $A \in SL_2(\mathbb{R})$, and let $\phi_A \in PSL_2(\mathbb{R}) = \mathrm{Isom}^+(\mathbb{H}^2)$ be the corresponding Möbius transformation. Prove that ϕ_A is:

   (a) hyperbolic if $|\mathrm{tr}\, A| > 2$;

   (b) elliptic if $|\mathrm{tr}\, A| < 2$;

   (c) parabolic or the identity if $|\mathrm{tr}\, A| = 2$.

4. Let Γ be a Fuchsian group. Prove that the centraliser of any non-trivial element of Γ is cyclic.

5. Let $\phi \in \mathrm{Isom}^+(\mathbb{H}^2)$ be a hyperbolic isometry. Prove that, after possibly replacing ϕ by its inverse, the endpoints $\xi_+, \xi_- \in \partial\mathbb{H}^2$ of the axis of ϕ have the following property: for any open subset $U \ni \xi_+$ and closed subset $V \not\ni \xi_-$ of $\partial\mathbb{H}^2$,

$$\phi^n(V) \subseteq U$$

for all sufficiently large n. [We say that ϕ acts with *north–south dynamics* on $\partial\mathbb{H}^2$.] Deduce that there is no metric on $\partial\mathbb{H}^2$ that is both compatible with the usual topology and invariant under the action of $\mathrm{Isom}^+(\mathbb{H}^2)$.

6. Let Γ be a Fuchsian group containing at least one hyperbolic element. The *limit set* of Γ, denoted by $\Lambda\Gamma$, is defined to be the closure in $\partial\mathbb{H}^2$ of the set of endpoints of axes of hyperbolic elements of Γ.

   (a) Prove that the natural action of Γ on $\partial\mathbb{H}^2$ preserves $\Lambda\Gamma$.

   (b) Show that either there is a point $\eta \in \partial\mathbb{H}^2$ fixed by every element of Γ, or every orbit of the action of Γ on $\Lambda\Gamma$ is dense in $\Lambda\Gamma$.

7. Let $p \leq q \leq r$ be positive integers, and consider the triangle group $\Gamma(p, q, r)$. If $1/p + 1/q + 1/r = 1$, prove that $\Gamma(p, q, r)$ acts compactly and properly discontinuously by isometries on the Euclidean plane. List all possibilities for p, q, r and draw pictures of the corresponding tilings.

8. (a) Let G be a group generated by elements a and b, acting on a set X. Suppose that there are disjoint subsets $A_+, A_-, B_+, B_- \subseteq X$ so that

$$a(A_+ \cup B_+ \cup B_-) \subseteq A_+ \quad , \quad b(B_+ \cup A_+ \cup A_-) \subseteq B_+$$
$$a^{-1}(A_- \cup B_+ \cup B_-) \subseteq A_- \quad , \quad b^{-1}(B_- \cup A_+ \cup A_-) \subseteq B_- \,.$$

   Show that every non-trivial reduced word in a and b moves a point in X. Deduce that G is freely generated by a and b. [This result is called *Klein's ping-pong lemma.*]

(b) Consider the Möbius transformations

$$\alpha(z) = z + 2\,,\ \beta(z) = \frac{z}{1 - 2z}\,.$$

Prove that the subgroup of $PSL_2(\mathbb{R})$ generated by α and β is free.

9. Consider the geodesic triangle Δ in the upper half-plane model of $\mathbb{H}^2$ with vertices at the points, $0, 1, \infty$. Compute the hyperbolic diameter of the largest semicircle inscribed in Δ. [Triangles in $\mathbb{H}^2$ with vertices on the boundary are called *ideal*.] Deduce the smallest possible δ such that $\mathbb{H}^2$ is δ-hyperbolic.